

# Quantizations of $\mathbb{R}$

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## Abstract

Quantum real numbers are proposed by performing a quantum deformation of the standard real numbers  $\mathbb{R}$ . We start with the  $q$ -deformed Heisenberg algebra  $\hat{\mathcal{L}}_q$  which is obtained by the Moyal  $*$ -deformation of the Heisenberg algebra generated by  $a$  and  $a^\dagger$ . By representing  $\hat{\mathcal{L}}_q$  as the algebras of  $q$ -differentiable functions, we derive quantum real lines from the base spaces of these functional algebras. We find that these quantum lines are discrete spaces. In particular, for the case with  $q = e^{2\pi i \frac{1}{N}}$ , the quantum real line is composed of fuzzy, i.e., fluctuating points and nontrivial infinitesimal structure appears around every standard real number.

## 1 Introduction

The long-standing problem, quantization of gravity theory, undoubtedly requires some fundamental modifications of the standard theory of geometry. By taking quantum effects into account, the concepts of, e.g., point, line, and so on should be reconsidered: (i) What is a point? Is it just a localized object of null size? (ii) What is a line? Is it just a one-dimensional continuous object? In other words, we should ask what quantum real number  $\mathbb{R}$  is. The aim of this paper is to challenge these essential problems.

As for the first problem, it can be expected that a point suffers quantum fluctuation and becomes something fuzzy, i.e., nonlocal object with some extra structure. Let us call such a quantum object a *quantum point*. As a set of quantum points, a *quantum real line* can be defined and, therefore, quantum real line must be fuzzy. Although we are much interested in geometries of higher dimensional quantum objects, this paper focuses on the quantum real line as the first step towards the quantum geometry.

A well-established example of quantum geometry is provided by the noncommutative geometry developed by Connes [1]. The noncommutative geometry has appeared

in the recent developments in string theory. Actually, noncommutative coordinates are used to describe D-brane. Another example is the quantum geometry based on quantum groups (see, for example, Ref.[2]). Let us observe briefly the above two approaches of quantization of geometry. Let  $M$  be the classical manifold or space which is to be quantized. The key process in both approaches is to define a commutative algebra  $A(M)$  on  $M$ . One then deforms  $A(M)$  to a noncommutative algebra  $\mathcal{A}(\mathcal{M})$  by some deformation procedure. In the Connes' noncommutative geometry, a functional algebra on  $M$  is taken as  $A(M)$ , the algebra to be deformed. As an extension of the Gelfand-Naimark theorem, which guarantees that geometrical data of  $M$  are extracted from  $A(M)$ , noncommutative geometry of the quantum space  $\mathcal{M}$  is derived from the noncommutative algebra  $\mathcal{A}(\mathcal{M})$ . On the other hand, in the quantum group approach,  $A(M)$  is provided by the coordinate ring  $A(G)$  of the group  $G$  which acts on the space  $M$ . The quantum geometry of the quantum space  $\mathcal{M}$  is derived from the quantum group  $A_q(G)$ .

Now, we are interested in the step from a commutative algebra  $A$  to a noncommutative algebra  $\mathcal{A}$ . In the quantum geometry via the quantum group, the quantum algebra  $A_q(G)$  is obtained from  $A(G)$  according to the general ideas of quantum group. In the Connes' approach, however, one can reach  $\mathcal{A}(\mathcal{M})$  by some procedures. A possible way is the Moyal deformation quantization, i.e., as a multiplication structure of  $\mathcal{A}(\mathcal{M})$ , a noncommutative Moyal  $*$ -product is used instead of the standard commutative product “.” in  $A(M)$ . Let us explain in more detail. Let  $\gamma$  be the deformation parameter which brings  $A(M)$  to  $\mathcal{A}(\mathcal{M})$ . The algebra  $\mathcal{A}(\mathcal{M})$  possesses the multiplication  $*_\gamma$  which is associative but noncommutative, i.e., for functions  $f, g, h \in \mathcal{A}(\mathcal{M})$ ,  $(f *_\gamma g) *_\gamma h = f *_\gamma (g *_\gamma h)$  and  $f *_\gamma g \neq g *_\gamma f$ . Note here that elements of  $\mathcal{A}(\mathcal{M})$  are the same as those of  $A(M)$ , and the only difference between  $A$  and  $\mathcal{A}$  is in the multiplication structure. Since geometrical data of  $\mathcal{M}$  are deduced from  $\mathcal{A}(\mathcal{M})$ ,  $\mathcal{M}$  is regarded as a noncommutative manifold. In particular, therefore, the coordinates in  $\mathcal{M}$  which are elements of  $\mathcal{A}(\mathcal{M})$  do not commute because of  $*_\gamma$ . By taking the limit  $\gamma \rightarrow 0$ ,  $\mathcal{A}(\mathcal{M})$  reduces to  $A(M)$  and, therefore,  $\mathcal{M} \xrightarrow{\gamma \rightarrow 0} M$ .

In this paper, we will propose quantum real numbers and show the geometries of quantum real lines guided partly by the above strategy and partly by Ref.[3]. The algebra with which we will start is the Heisenberg algebra  $\mathcal{L}_H$  generated by the operators  $a$  and  $a^\dagger$ , since  $\mathcal{L}_H$  is equivalent to the algebra of differentiable functions

on the real numbers  $\mathbb{R}$ . In order to obtain quantum real numbers, we will deform  $\mathcal{L}_H$  in Section 2 and reach the  $q$ -deformed Heisenberg algebra  $\hat{\mathcal{L}}_q$ . The quantization from  $\mathcal{L}_H$  to  $\hat{\mathcal{L}}_q$  is performed according to the procedure developed in [3]. Namely,  $\hat{\mathcal{L}}_q$  is introduced as  $\hat{\mathcal{L}}_q := \mathcal{L}_H \times \hat{\mathcal{A}}(\mathcal{T}_1)$ . Here,  $\hat{\mathcal{A}}(\mathcal{T}_1)$  is the internal sector and, explicitly, an operator algebra on the one-dimensional torus  $\mathcal{T}_1$ . The operator algebra  $\hat{\mathcal{A}}(\mathcal{T}_1)$  is derived from the noncommutative functional algebra  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  on the two-dimensional torus  $\mathcal{T}_2$ . In Section 3, we will represent  $\hat{\mathcal{L}}_q$  as the algebra of  $q$ -differentiable functions. We will, in Section 3.1, look at the case where  $q$  is not a root of unity and construct the algebra  $\mathcal{D}^q(\mathfrak{R})$  of  $q$ -differentiable functions on the base space  $\mathfrak{R}$ . Section 3.2 looks at the case where  $q$  is the  $N$ -th root of unity and the algebra  $\mathcal{D}_N^q(\mathfrak{R}_N)$  is built. The section 4 is the main part of this paper, where the structures of the base spaces  $\mathfrak{R}$  and  $\mathfrak{R}_N$  are investigated through the algebras  $\mathcal{D}^q(\mathfrak{R})$  and  $\mathcal{D}_N^q(\mathfrak{R}_M)$ , respectively. We will finally propose, in Section 4.1, the quantum real line  $\mathbb{R}_D$  via the base space  $\mathfrak{R}$ . and, in Section 4.2, the quantum real lines  $\mathbb{R}_q^N$  and  ${}^\forall\mathbb{R}$  via  $\mathfrak{R}_N$ .

We will further discuss geometrical features of these lines and show the followings; the line  $\mathbb{R}_D$  is a discrete space which is composed of an infinite number of *standard* points. Namely, the minimal length appears in  $\mathbb{R}_D$ . The lines  $\mathbb{R}_q^N$  and  ${}^\forall\mathbb{R}$  are also discrete spaces. However, each point in these lines is not a standard point but a *fuzzy* point. Indeed, we will give a definition of a “wave function of a point” and finds that wave functions of fuzzy points are not localized but spread. It should be remarked that the standard real line  $\mathbb{R}$  appears as a subset, i.e.,  $\mathbb{R} \subset \mathbb{R}_q^N, {}^\forall\mathbb{R}$ . The spaces  $\mathbb{R}_q^N \setminus \mathbb{R}$  and  ${}^\forall\mathbb{R} \setminus \mathbb{R}$  are called *infinitesimal structures* in the sense that the quotient spaces consist of points living between  $x$  and  $x + \epsilon$  for  ${}^\forall x \in \mathbb{R}$  and  $\epsilon \in \mathbb{R}$  being an infinitesimal.

Concluding remarks, especially on the features in the limit  $q \rightarrow 1$ , and future problems are addressed in Section 5.

## 2 Deformation of the Heisenberg algebra

We start with a deformation of the Heisenberg algebra  $\mathcal{L}_H$ , which is the operator algebra generated by the two generators  $a$  and  $a^\dagger$  satisfying the defining relation

$$aa^\dagger - a^\dagger a = 1. \quad (1)$$

Although some deformations of  $\mathcal{L}_H$  have been discussed in lots of works, let us follow the deformation procedure developed in Ref.[3], where the origin of quantum effects is given manifestly.

The first and essential step is the introduction of the algebra  $\mathcal{A}(\mathcal{T}_2)$  of functions on the two-dimensional torus  $\mathcal{T}_2$ . Let  $\theta$  and  $\tau$  be the variables on  $\mathcal{T}_2$ . Then, the generators of  $\mathcal{A}(\mathcal{T}_2)$  are written as

$$U := e^{i\theta}, \quad V := e^{i\tau}, \quad (2)$$

i.e.,  $\mathcal{A}(\mathcal{T}_2) \ni f = f(U, V)$ . Upon interpreting  $\mathcal{A}(\mathcal{T}_2)$  as the internal sector, we will take the product  $\mathcal{L}_H \times \mathcal{A}(\mathcal{T}_2)$  later in order to define our deformed Heisenberg algebra. Before doing this, we have to introduce a multiplication into  $\mathcal{A}(\mathcal{T}_2)$ .

The most familiar and simplest one is the standard local product “.” and let  $\mathcal{A}(\mathcal{T}_2; \cdot)$  be the algebra with  $\cdot$ . Obviously,  $\mathcal{A}(\mathcal{T}_2; \cdot)$  is a closed and commutative algebra, i.e., for  $\forall f(U, V), g(U, V) \in \mathcal{A}(\mathcal{T}_2; \cdot)$ ,  $f \cdot g = g \cdot f \in \mathcal{A}(\mathcal{T}_2; \cdot)$ . In this case, however, one *cannot* expect  $\mathcal{L}_H \times \mathcal{A}(\mathcal{T}_2; \cdot)$  to be a nontrivial deformation of the Heisenberg algebra. Indeed, one concludes with the isomorphism  $\mathcal{L}_H \times \mathcal{A}(\mathcal{T}_2; \cdot) \cong \mathcal{L}_H$ , since the internal sector  $\mathcal{A}(\mathcal{T}_2; \cdot)$  is factorized completely from the algebra  $\mathcal{L}_H$ .

Another possible product is the, so-called, Moyal product  $*_\gamma$  with a deformation parameter  $\gamma$ . Let  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  be the algebra endowed with  $*_\gamma$ . Here, the Moyal product  $*_\gamma$  is given explicitly as [3]

$$*_\gamma = \sum_{n=0}^{\infty} \frac{\left(\frac{-i\gamma}{2}\right)^n}{n!} \sum_{k=0}^n \binom{n}{k} (-)^k \overleftarrow{\partial_\theta^{n-k} \partial_\tau^k} \cdot \overrightarrow{\partial_\theta^k \partial_\tau^{n-k}}. \quad (3)$$

Since the product (3) is nonlocal, the algebra  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  is no longer commutative algebra but a noncommutative algebra with the defining commutation relations as

$$U *_\gamma V = e^{i\gamma} V *_\gamma U, \quad (4)$$

$$\theta *_\gamma \tau - \tau *_\gamma \theta = -i\gamma. \quad (5)$$

The commutation relations in (4, 5) indicate that the base space  $\mathcal{T}_2$  of  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  is the so-called noncommutative torus. into the Heisenberg algebra. Let us define

$$\mathcal{L}_H^q = \mathcal{L}_H \times \mathcal{A}(\mathcal{T}_2; *_\gamma). \quad (6)$$

One can now expect that  $\mathcal{L}_H^q$  provides nontrivial deformations of  $\mathcal{L}_H$ . Let us explain in more detail. We define the algebra  $\mathcal{L}_H^q$  explicitly by giving the generators as

$$a_\theta := aU, \quad a_\tau^\dagger := a^\dagger V. \quad (7)$$

Then, the algebra  $\mathcal{L}_H^q$  is an operator algebra spanned by the functions of operators  $a_\theta, a_\tau^\dagger$ , where the operator nature originates only in the sector  $\mathcal{L}_H$ . The multiplication is introduced into  $\mathcal{L}_H^q$  as follows; the standard operator product is assumed in the sector  $\mathcal{L}_H$ , while the Moyal product (3) is used for the internal sector  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$ . By making use of the commutation relations given in eqs.(1) and (4), one immediately obtains the deformed commutation relation between the generators  $a_\theta, a_\tau^\dagger$  as

$$a_\theta *_\gamma a_\tau^\dagger - q a_\tau^\dagger *_\gamma a_\theta = U *_\gamma V, \quad (8)$$

where  $q$  is the deformation parameter given by

$$q = e^{i\gamma}. \quad (9)$$

Note here that, in the limit  $\gamma \rightarrow 0$ , the deformed algebra  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  reduces to the commutative algebra  $\mathcal{A}(\mathcal{T}_2; \cdot)$ . Therefore, as discussed in the preceding paragraph, the deformed algebra  $\mathcal{L}_H^q$  reduces to the original Heisenberg algebra  $\mathcal{L}_H$ , i.e.,  $\mathcal{L}_H^q \xrightarrow{\gamma \rightarrow 0} \mathcal{L}_H$ .

Having obtained the operator algebra  $\mathcal{L}_H^q$ , our interests are in a representation of the algebra and the geometry of the base space on which  $\mathcal{L}_H^q$  acts. According to the Gelfand-Naimark theorem and the strategy of Connes [1], the geometrical structures of the space will be extracted from  $\mathcal{L}_H^q$ . These investigations are the tasks in the following sections.

However, a formulation with the Moyal product is not always convenient to handle. Let us, therefore, derive another 1-deformed algebra without the Moyal product from  $\mathcal{L}_H^q$ . In the derivation, we are required that the  $q$ -deformed effects originated from the relation (8) should be preserved. Recalling that, in  $\mathcal{L}_H^q$ , the  $q$ -deformations come from the noncommutativity of  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$ , our problem is how to derive the noncommutativity without using the Moyal product. A solution is to use the operator formalism as follows: The functional algebra  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  is changed into the operator algebra  $\hat{\mathcal{A}}(\mathcal{T}_1)$  where  $\mathcal{T}_1$  is a one-dimensional configuration space reduced from  $\mathcal{T}_2$  regarded as the phase space. Namely,  $\hat{\mathcal{A}}(\mathcal{T}_1)$  acts on the space of functions on  $\mathcal{T}_1$ . We

introduce the operator-valued generators  $\hat{U}, \hat{V}$  of  $\hat{\mathcal{A}}(\mathcal{T}_1)$  so that they should satisfy the commutation relation

$$\hat{U}\hat{V} = q\hat{V}\hat{U} \quad (10)$$

instead of (4). Upon writing these generators as  $\hat{U} = e^{i\hat{\theta}}, \hat{V} = e^{i\hat{\tau}}$ , one easily finds that the requirement given in (10) follows from the commutation relation

$$\hat{\theta}\hat{\tau} - \hat{\tau}\hat{\theta} = -i\gamma, \quad (11)$$

which is compatible with the relation (5). Then, a representation of the algebra  $\hat{\mathcal{A}}(\mathcal{T}_1)$  is performed by imposing the polarization,

$$\hat{\theta} = \theta, \quad \hat{\tau} = i\gamma \frac{d}{d\theta} := i\gamma \partial_\theta. \quad (12)$$

Therefore, the configuration space  $\mathcal{T}_1$  is one-dimensional torus parameterized by  $\theta, 0 \leq \theta < 2\pi$ .

Now, we have obtained the internal sector  $\hat{\mathcal{A}}(\mathcal{T}_1)$ . Let us define the  $q$ -deformed Heisenberg algebra  $\hat{\mathcal{L}}_q$  as

$$\hat{\mathcal{L}}_q := \mathcal{L}_H \times \hat{\mathcal{A}}(\mathcal{T}_1). \quad (13)$$

What we should make explicit are the generators of  $\hat{\mathcal{L}}_q$  and the defining commutation relation between them. Let  $\hat{a}_q, \hat{a}_q^\dagger$  be the generators and define them by the following substitutions,

$$a\hat{U} \longmapsto \hat{a}_q, \quad e^{i\theta} a^\dagger \hat{V} \longmapsto \hat{a}_q^\dagger. \quad (14)$$

The defining relation of  $\hat{\mathcal{L}}_q$  is finally obtained as

$$\hat{a}_q \hat{a}_q^\dagger - q \hat{a}_q^\dagger \hat{a}_q = q^{i\partial_\theta}. \quad (15)$$

### 3 Deformations of algebra of functions on $\mathbb{R}$

Recall first that a possible representation of the Heisenberg algebra  $\mathcal{L}_H$  is realized by the replacements  $a \rightarrow \partial_x$  and  $a^\dagger \rightarrow \hat{x}$  where  $\partial_x f(x) = \frac{d}{dx} f(x)$  and  $\hat{x} f(x) = x f(x)$  for differentiable functions  $f(x)$  on  $\mathbb{R}$ . We denote the operator algebra generated by  $\hat{x}$  and  $\partial_x$  as  $\hat{\mathcal{O}}(\mathbb{R})$ . Let us adopt similar replacements to the  $q$ -deformed Heisenberg algebra  $\hat{\mathcal{L}}_q$  and obtain the  $q$ -deformed operator algebra  $\hat{\mathcal{O}}(\mathfrak{R})$  which acts on the space of functions on the space  $\mathfrak{R}$ . Upon taking the relationship between  $\mathcal{L}_H$  and  $\mathbb{R}$  into account, we expect that  $\mathfrak{R}$  is raised to a  $q$ -deformation of  $\mathbb{R}$ . We should give a

comment here on the base space  $\mathfrak{R}$ . Although, according to our construction of  $\hat{\mathcal{L}}_q$ , it is natural to regard  $\mathfrak{R}$  as the tensor product of two spaces, i.e., the space for  $\mathcal{L}_H$  and just  $\mathcal{T}_1$ , we will consider  $\mathfrak{R}$  to be the one-dimensional space parameterized by  $\vartheta$  as

$$\mathfrak{R} = \{\vartheta \mid -\infty \leq \vartheta \leq \infty\}, \quad (16)$$

and find such a representation.

Let  $\hat{\chi}$  and  $D_q$  be the generators of  $\hat{\mathcal{O}}(\mathfrak{R})$ . Namely, upon the replacements

$$\hat{a}_q \longrightarrow D_q, \quad \hat{a}_q^\dagger \longrightarrow \hat{\chi}, \quad (17)$$

$\hat{\chi}$  and  $D_q$  are identified, respectively, with the  $q$ -deformed coordinate operator and the  $q$ -deformed differential operator. Let  $\mathcal{D}^q(\mathfrak{R})$  be the algebra of functions on which the action of  $\hat{\mathcal{O}}(\mathfrak{R})$  is defined. Geometrical structure of  $\mathfrak{R}$  is deduced from  $\mathcal{D}^q(\mathfrak{R})$ . In the following subsections, we are going to investigate  $\mathcal{D}^q(\mathfrak{R})$  for two cases; Section 3.1 looks at the case with a generic  $q$  and the case with  $q$  at a root of unity is studied in Section 3.2.

### 3.1 The case where $q$ is generic

Let us suppose first that the deformation parameter  $q$  is not a root of unity. Upon the substitutions (17), the defining relation (15) reads the commutation relation between  $D_q$  and  $\hat{\chi}$  as

$$D_q \hat{\chi} - q \hat{\chi} D_q = q^{i\partial_\vartheta}. \quad (18)$$

The explicit expressions of these operators are given by

$$\hat{\chi} = \frac{1 + q^{i\partial_\vartheta}}{1 + q^{-1}} e^{i\vartheta}, \quad D_q = \frac{q^{-i\partial_\vartheta} - 1}{e^{i\vartheta}(q - 1)}. \quad (19)$$

Let us use the variable  $x_\vartheta$  instead of  $\vartheta$  and redefine the base space  $\mathfrak{R}$  as

$$\mathfrak{R} = \{x_\vartheta \mid x_\vartheta := e^{i\vartheta}, -\infty \leq \vartheta \leq \infty\}. \quad (20)$$

One then constructs the functional algebra  $\mathcal{D}^q(\mathfrak{R})$  as

$$\mathcal{D}^q(\mathfrak{R}) = \{f(x_\vartheta) \mid x_\vartheta \in \mathfrak{R}, \}, \quad (21)$$

where the multiplication in  $\mathcal{D}^q(\mathfrak{R})$  is supposed to be the standard pointwise product “.”. The complete basis of  $\mathcal{D}^q(\mathfrak{R})$  are given by

$$\varpi := \{x_\vartheta^n \mid n \in \mathbb{Z}\} \quad (22)$$

and, therefore,  $\forall f(x_\vartheta) \in \mathcal{D}^q(\mathfrak{R})$  is expanded as

$$f(x_\vartheta) = \sum_{n \in \mathbf{Z}} f_n x_\vartheta^n. \quad (23)$$

The actions of the operators  $\hat{\chi}$  and  $D_q$  on  $\mathcal{D}^q(\mathfrak{R})$  are calculated explicitly by (19) as

$$\hat{\chi} f(x_\vartheta) = x_\vartheta \frac{f(x_\vartheta) + q^{-1} f(x_\vartheta q^{-1})}{1 + q^{-1}}, \quad D_q f(x_\vartheta) = \frac{f(x_\vartheta q) - f(x_\vartheta)}{x_\vartheta(q - 1)}. \quad (24)$$

Note that, by taking the limit  $q \rightarrow 1$  together with  $x_\vartheta \rightarrow x$ ,  $\hat{\chi}$  becomes the standard coordinate operator  $\hat{x}$  and  $D_q$  reduces to the standard differential operator  $\partial_x$ , i.e.,  $\hat{\mathcal{O}}(\mathfrak{R}) \ni \hat{\chi}, D_q \xrightarrow{q \rightarrow 1} \hat{x}, \partial_x \in \hat{\mathcal{O}}(\mathbb{R})$ . Furthermore, one shows that  $D_q$  satisfies the following properties,

$$D_q \mathbf{1} = 0, \quad \text{for the identity } \mathbf{1} \in \mathcal{D}^q(\mathfrak{R}), \quad (25)$$

$$D_q(f \cdot g) = (D_q f(x_\vartheta)) g(x_\vartheta) + f(q x_\vartheta) (D_q g(x_\vartheta)). \quad (26)$$

The first equation (25) indicates that  $D_q$  vanishes constants in  $\mathcal{D}^q(\mathfrak{R})$ , and (26) shows the deformed Leibniz rule.

It is interesting to observe finally that the  $q$ -deformed exterior derivative  $d_q$  can be defined through

$$d_q f = d_q x_\vartheta D_q f(x_\vartheta). \quad (27)$$

One immediately finds that

$$d_q(f \cdot g) = (d_q f) \cdot g + f \cdot (d_q g) \quad (28)$$

holds if and only if another noncommutativity

$$d_q x_\vartheta f(q x_\vartheta) = f(x_\vartheta) d_q x_\vartheta \quad (29)$$

is satisfied. Thus we have found the “noncommutative” structures shown in eqs.(26, 29). This is the reason why we will refer to  $\mathcal{D}^q(\mathfrak{R})$  as “noncommutative algebra”, although any elements in  $\mathcal{D}^q(\mathfrak{R})$  are mutually commutable. In other words, the noncommutative natures originate just from the  $q$ -differential operator.

### 3.2 The case where $q$ is a root of unity

Let us turn to the case where the deformation parameter  $q$  is a root of unity, i.e., for a positive integer  $N$ ,

$$q = e^{i \frac{2\pi}{N}}. \quad (30)$$



We will use the variable  $y_\vartheta$  in this case and rewrite the base space  $\mathfrak{R}$  as

$$\mathfrak{R}_N = \{ y_\vartheta \mid y_\vartheta := e^{i\vartheta}, -\infty \leq \vartheta \leq \infty \}. \quad (31)$$

Let  $\mathcal{D}_N^q(\mathfrak{R}_N)$  be the algebra of functions on which the  $q$ -coordinate operator  $\hat{\chi}$  and the  $q$ -differential operator  $D_q$  with respect to  $y_\vartheta$  are defined. The basis of  $\mathcal{D}_N^q(\mathfrak{R}_N)$  are then

$$\varpi_N := \{ y_\vartheta^n \mid n \in \mathbb{Z} \}, \quad (32)$$

i.e., any function  $f(y_\vartheta) \in \mathcal{D}_M^q(\mathfrak{R}_N)$  is expanded as

$$f(y_\vartheta) = \sum_{n \in \mathbb{Z}} \alpha_n y_\vartheta^n. \quad (33)$$

Before going to the investigations of  $\mathcal{D}_N^q(\mathfrak{R}_N)$ , recall that representations of a quantum group  $G_q$  with  $q$  at a root of unity are drastically different from those of the quantum group  $G_q$  with generic  $q$  and of the classical group  $G$ . We will see below that the structure of  $\mathcal{D}_N^q(\mathfrak{R}_N)$  is also different from that of  $\mathcal{D}^q(\mathfrak{R})$  completely.

In order to observe the structure of  $\mathcal{D}_N^q(\mathfrak{R}_N)$ , one should start with the actions of the operators  $\hat{\chi}$  and  $D_q$  defined in (19). In particular, the actions of  $D_q$  on the basis  $\varpi_N$  are to be remarked. Actually, upon using  $[N] = 0$ , one finds

$$\begin{aligned} D_q y_\vartheta^{kN+r} &= [r] y_\vartheta^{kN+r-1}, \quad \text{for } r \neq 0, \\ D_q y_\vartheta^{kN} &= 0. \end{aligned} \quad (34)$$

Thus,  $y_\vartheta^N$  behaves as a constant with respect to  $D_q$ . Further, one finds easily that the operator  $D_q^N$  on  $\mathcal{D}_N^q(\mathfrak{R}_N)$  is null, i.e.,  $D_q^N f(y_\vartheta) = 0, \forall f(y_\vartheta) \in \mathcal{D}_N^q$ . However, upon defining another operator  $\partial$  by  $\partial := D_q^N / [N]!$ ,  $\partial$  acts on  $\varpi_N$  as

$$\begin{aligned} \partial y_\vartheta^{kN+r} &= k y_\vartheta^{(k-1)N+r}, \quad \text{for } k \in \mathbb{Z}, \\ \partial y_\vartheta^r &= 0, \quad \text{for } r \leq N-1, \end{aligned} \quad (35)$$

where use has been made of the relation  $[kN]/[N] = k$ . One should notice that, with respect to  $\partial$ ,  $y_\vartheta^r$  for  $1 \leq r \leq N-1$  are regarded as constants in  $\mathcal{D}_N^q(\mathfrak{R}_N)$ . The facts derived from (34) and (35) are:

1. These operators  $D_q$  and  $\partial$  commute with each other.

2. The variables  $y_\vartheta^{\pm r}$  ( $r = 1, \dots, N-1$ ) and  $y_\vartheta^{kN}$ ,  $k \in \mathbf{Z}$  are independent of each other.

Therefore, one should require that two independent variables  $y_\vartheta$  and  $y_\vartheta^N$  are needed for the perfect definition of the algebra  $\mathcal{D}_N^q(\mathfrak{R}_N)$  and further that two independent ( $q$ -deformed) differential operators  $D_q$  and  $\partial$  should be introduced.

Guided by the above observations, let us investigate the structure of  $\mathcal{D}_N^q(\mathfrak{R}_N)$  in detail. To this end, we introduce the map  $\pi : \mathcal{D}_N^q(\mathfrak{R}_N) \rightarrow \mathcal{D}(\mathcal{R}) \otimes \mathcal{D}_I^q(\mathcal{S}_1)$  by

$$\pi(y_\vartheta^{kN+r}) = x^k \xi^r. \quad (36)$$

Namely, for a function  $f(y_\vartheta) \in \mathcal{D}_N^q(\mathfrak{R}_N)$ ,

$$\pi(f(y_\vartheta)) = \lambda(x) \otimes \psi(\xi), \quad (37)$$

$$\lambda \in \mathcal{D}, \quad x \in \mathcal{R} \quad \psi \in \mathcal{D}_I^q, \quad \xi \in \mathcal{S}_1,$$

where  $k \in \mathbf{Z}$  and  $r$  is in the region  $\mathcal{I}$  as

$$r \in \mathcal{I} := \begin{cases} \{-p, -p+1, \dots, p-1, p\}, & \text{for } N = 2p+1, \\ \{-p+1, \dots, p-1, p\}, & \text{for } N = 2p \end{cases}. \quad (38)$$

The basis  $\varpi_N$  given in (32) is also factorized as  $\varpi_N \rightarrow \varpi_{out} \otimes \varpi_{int}$  where  $\varpi_{out} := \{x^k \mid k \in \mathbf{Z}\}$  and  $\varpi_{int} := \{\xi^r \mid r \in \mathcal{I}\}$ . Namely, functions  $\lambda(x) \in \mathcal{D}(\mathcal{R})$  and  $\psi(\xi) \in \mathcal{D}_I^q(\mathcal{S}_1)$  are expanded as

$$\lambda(x) = \sum_{k \in \mathbf{Z}} \lambda_k x^k, \quad \psi(\xi) = \sum_{r \in \mathcal{I}} \psi_r \xi^r. \quad (39)$$

Let us go back to the ( $q$ -deformed) differential operators  $\partial$ ,  $D_q$  and discuss their factorizations induced from  $\pi$ . The equations shown in (34,35) suggest that the operators  $\partial$  and  $D_q$  are factorized as

$$\partial \longmapsto \partial_x \otimes id, \quad D_q \longmapsto id \otimes \nabla_q, \quad (40)$$

i.e.,  $\partial_x$  and  $\nabla_q$  are ( $q$ -)differential operators defined, respectively, on  $\mathcal{D}(\mathcal{R})$  and  $\mathcal{D}_I^q(\mathcal{S}_1)$  such as

$$\partial_x \lambda(x) = \frac{d}{dx} \lambda(x), \quad \nabla_q \psi(\xi) = \frac{\psi(q\xi) - \psi(\xi)}{\xi(q-1)}. \quad (41)$$

In order to make our story complete, we have to determine the multiplication structures in both sectors. The first sector  $\mathcal{D}(\mathcal{R})$  can be endowed with the standard

multiplication “ $\cdot$ ” and, with the multiplication,  $\mathcal{D}(\mathcal{R})$  is the standard algebra of differentiable functions on  $\mathcal{R}$ . On the contrary, the sector  $\mathcal{D}_I^q(\mathcal{S}_1)$  cannot possess “ $\cdot$ ”, since the sector does not close with respect to the product. Actually, for  $r, r' \in \mathcal{I}$ , the value  $s$  such as  $\xi^r \cdot \xi^{r'} = \xi^s$  is not always in  $\mathcal{I}$ . One can make  $\mathcal{D}_I^q(\mathcal{S}_1)$  a closed algebra by introducing the mapping  $\mu : \mathcal{D}_I^q(\mathcal{S}_1) \rightarrow \hat{\mathcal{D}}^q(M)$ . Here  $\hat{\mathcal{D}}^q(M)$  is the algebra of  $N \times N$  matrices and is spanned by the basis

$$\hat{\xi}^r = \mu(\xi^r) = \begin{pmatrix} & I_{N-r} \\ I_r & \end{pmatrix}, \quad r \in \mathcal{I}, \quad (42)$$

where  $I_n$  is the  $n \times n$  unit matrix with the convention  $I_{N+n} = I_n$ . Notice that the algebra  $\hat{\mathcal{D}}^q(M)$  is closed with the standard matrix multiplication, and  $\hat{\omega}_I = \{\hat{\xi}^r \mid r \in \mathcal{I}\}$  is a complete basis. Since the mapping  $\mu$  is an isomorphism between  $\mathcal{D}_I^q(\mathcal{S}_1)$  and  $\hat{\mathcal{D}}^q(M)$ , the multiplication in  $\mathcal{D}_I^q(\mathcal{S}_1)$  is introduced as follows; upon denoting the multiplication as  $\odot$ , the product of two functions  $\psi, \psi' \in \mathcal{D}_I^q(\mathcal{S}_1)$  is defined by

$$\psi(\xi) \odot \psi'(\xi) := \mu^{-1}(\mu(\psi)\mu(\psi')). \quad (43)$$

Now the  $q$ -deformed algebra  $\mathcal{D}_N^q(\mathfrak{R}_N)$  of  $q$ -differentiable functions on  $\mathfrak{R}_N$  has been at hand, i.e.,

$$\mathcal{D}_N^q(\mathfrak{R}_N) \cong \mathcal{D}(\mathcal{R}) \otimes \mathcal{D}_I^q(\mathcal{S}_1; \odot), \quad (44)$$

where we have rewritten the internal sector as  $\mathcal{D}_I^q(\mathcal{S}_1; \odot)$  in order to make the multiplication structure explicit. Thus, when  $q$ -differential structure is considered, the algebra of functions on  $\mathfrak{R}_N$  is necessarily decomposed into two sectors: The first sector  $\mathcal{D}(\mathcal{R})$  is the standard (undeformed) algebra of differentiable functions, while the second sector  $\mathcal{D}_I^q(\mathcal{S}_1; \odot)$ , which is called the internal sector, is an algebra of  $q$ -differentiable functions with the multiplication “ $\odot$ ”. At the same time, the base space  $\mathfrak{R}_N$  has the tensor structure

$$\mathfrak{R}_N = \mathcal{R} \otimes \mathcal{S}_1 \ni (x, \xi). \quad (45)$$

Here, the space  $\mathcal{S}_1$  is regarded as the internal space attached to every point of the external space  $\mathcal{R}$ . The geometrical structure of  $\mathfrak{R}_N$  will be discussed in the next section.

## 4 Quantum real lines

We have obtained the algebras of  $q$ -differentiable functions: When the deformation parameter  $q$  is *not* a root of unity, we have  $\mathcal{D}^q(\mathfrak{R})$  on the base space  $\mathfrak{R}$ . In the case where  $q$  is at the  $N$ -th root of unity, the algebra is  $\mathcal{D}_N^q(\mathfrak{R}_N)$  having the structure shown in (44) and the base space  $\mathfrak{R}_N$  is expected to have the form in (45).

Let us go ahead to the main task of our program, i.e., to investigate geometrical structures of  $\mathfrak{R}$  and  $\mathfrak{R}_N$  and to propose quantum real lines. This final step is performed via the algebras  $\mathcal{D}^q(\mathfrak{R})$  and  $\mathcal{D}_N^q(\mathfrak{R}_N)$ . The quantum real line  $\mathbb{R}_D$  will be derived from  $\mathfrak{R}$  in Section 4.1, and the lines  $\mathbb{R}_q^N$  and  ${}^v\mathbb{R}$  will appear through  $\mathfrak{R}_N$  in Section 4.2.

### 4.1 Quantum real line $\mathbb{R}_D$ ; $q$ is not a root of unity

Notice first that, as we have seen in Section 3.1, the  $q$ -differential structure of the algebra  $\mathcal{D}^q(\mathfrak{R})$  appears as the difference structure brought by  $D_q$ . It is, therefore, expected that the base space  $\mathfrak{R}$  has some discrete structure, although the variable  $x_\vartheta$  has been treated as a continuous variable so far.

One can find that the discrete structure appears in  $\mathfrak{R}$  by choosing a point  $x_{\vartheta_0} \in \mathfrak{R}$  arbitrarily. Upon the choice, the algebra  $\mathcal{D}^q(\mathfrak{R})$  reduces to the subalgebra  $\mathcal{D}^q(\mathfrak{R}_0)$  where

$$\mathfrak{R}_0 := \{x_{\vartheta_n} \mid \vartheta_n = \vartheta_0 + n\gamma, \ n \in \mathbf{Z}\}. \quad (46)$$

Recalling that  $x_\vartheta = e^{i\vartheta}$ , points in  $\mathfrak{R}_0$  cover  $S_1$  densely, since  $\gamma$  is now an irrational number. Then, the algebra  $\mathcal{D}^q(\mathfrak{R})$  is just the direct sum of  $\mathcal{D}^q(\mathfrak{R}_0)$  as

$$\mathcal{D}^q(\mathfrak{R}) = \bigoplus_{0 \leq \vartheta_0 < \gamma} \mathcal{D}^q(\mathfrak{R}_0). \quad (47)$$

One immediately deduces that the base space  $\mathfrak{R}$  is the union of all  $\mathfrak{R}_0$  as

$$\mathfrak{R} = \bigcup_{0 \leq \vartheta_0 < \gamma} \mathfrak{R}_0. \quad (48)$$

It is important to notice that all the subalgebras are equivalent independently of the point  $\vartheta_0$ . Therefore, one has only to pick up one subalgebra  $\mathcal{D}^q(\mathfrak{R}_0)$  and  $\mathfrak{R}_0$  as the base space, as far as the action of the operator algebra  $\widehat{\mathcal{O}}(\mathfrak{R})$  is considered. Now, the discrete space  $\mathfrak{R}_0$  can be regarded as a candidate of the quantum real line. What we

should emphasize is that every point in  $\mathfrak{R}_0$  is standard, i.e., *local* point of null size. Actually, the basis  $\varpi$  of  $\mathcal{D}^q(\mathfrak{R})$  given in (22) guarantees that the Dirac's  $\delta$ -function exists in  $\mathcal{D}^q(\mathfrak{R})$ . Therefore, the space  $\mathfrak{R}$  and the subspace  $\mathfrak{R}_0$  as well, consist of such standard points. This fact reads that a function  $f \in \mathcal{D}^q(\mathfrak{R}_0)$  takes its values just on the points  $x_{\vartheta_n} \in \mathfrak{R}_0$  and, at each point, the function has sharp peak.

Let us define the  $q$ -deformed real line via  $\mathfrak{R}_0$ . To do this, we introduce the new variable  $\chi$  through the relation

$$\chi_n = -i \log x_{\vartheta_n}, \quad (49)$$

and denote the space parameterized by  $\chi_n$  as  $\mathbb{R}_D$ , i.e.,

$$\mathbb{R}_D := \{\chi_n \mid n \in \mathbf{Z}\}. \quad (50)$$

In order to obtain geometrical data of  $\mathbb{R}_D$ , let us study  $\mathcal{D}^\gamma(\mathbb{R}_D)$ , the algebra of functions on  $\mathbb{R}_D$ . The difference operator  $\Delta_\gamma$  with respect to the variable  $\chi$  is induced from  $D_q$  as

$$\Delta_\gamma = \frac{\mathcal{K} - 1}{\gamma}, \quad (51)$$

where we have assumed the relation  $D_q = (D_q \chi) \Delta_\gamma$ . The operator  $\mathcal{K}$  in (51) is the shift operator such as  $\mathcal{K}f(\chi) = f(\chi + \gamma)$  and, therefore, the difference operator  $\Delta_\gamma$  acts on a function  $f \in \mathcal{D}^\gamma(\mathbb{R}_D)$  as

$$\Delta_\gamma f(\chi) = \frac{f(\chi + \gamma) - f(\chi)}{\gamma}. \quad (52)$$

Further, the key commutation relation between the coordinate  $\chi$  and  $\Delta_\gamma$  is calculated as

$$\chi \Delta_\gamma - \Delta_\gamma \chi = \mathcal{K}. \quad (53)$$

Notice that the functional algebra  $\mathcal{D}^\gamma(\mathbb{R}_D)$  is closed under the action of  $\Delta_\gamma$ , i.e., all the functions in  $\mathcal{D}^\gamma(\mathbb{R}_D)$  live just on  $\mathbb{R}_D$ . We are at the stage to propose the base space  $\mathbb{R}_D$  as the quantum real line in the case where the deformation parameter  $q$  is not a root of unity:

■ **Proposition 1** : Quantum Real Line  $\mathbb{R}_D$

*The quantum real line  $\mathbb{R}_D$  for the case with generic deformation parameter  $q$  is a discrete space composed of an infinite number of points  $\chi_n$  as*

$$\mathbb{R}_D = \{\chi_n \mid \chi_n = n\gamma + \chi_0, \quad n \in \mathbb{Z}\}. \quad (54)$$

It should be emphasized again that each point in  $\mathbb{R}_D$  is a localized object, i.e.,  $\chi_n$  are the standard points in the Euclidean sense. The quantum real line  $\mathbb{R}_D$  is depicted in Fig.1.

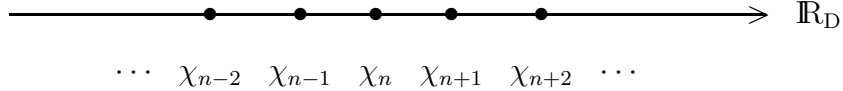


Fig.1 Quantum real line  $\mathbb{R}_D$

Thus, the quantum real line  $\mathbb{R}_D$  has the *minimal length*  $\gamma$ .

Let us finally look at the limit  $\gamma \rightarrow 0$ , i.e.,  $q \rightarrow 1$ . Since the interval between two adjacent points is  $\gamma$ , the quantum line  $\mathbb{R}_D$  becomes a continuous line. Further,  $\mathbb{R}_D$  becomes a smooth line, since the difference operator  $\Delta_\gamma$  reduces to the standard differential operator  $\partial_\chi$  with respect to the variable  $\chi$ . These observations indicate that, in the limit  $\gamma \rightarrow 0$ ,  $\chi$  can be regarded as the standard real number and, therefore,  $\mathbb{R}_D \xrightarrow{\gamma \rightarrow 0} \mathbb{R}$ .

## 4.2 Quantum real line ${}^\vee\mathbb{R}$ ; $q$ is a root of unity

Let us turn to the case where the deformation parameter  $q$  is the  $N$ -th root of unity. We will investigate the base space  $\mathfrak{R}_N$  via the algebra  $\mathcal{D}_N^q(\mathfrak{R}_N)$  and propose the quantum real line for this case. Recalling that the space  $\mathfrak{R}_N$  is isomorphic to the tensor product of two sectors  $\mathcal{R}$  and  $\mathcal{S}_1$  as shown in (45), we are going to investigate geometrical structures of these spaces  $\mathcal{R}$  and  $\mathcal{S}_1$  separately in terms of the algebras  $\mathcal{D}(\mathcal{R})$  and  $\mathcal{D}_1^q(\mathcal{S})$ .

We have found in Section 3.2 that the first sector  $\mathcal{D}(\mathcal{R})$  is just the standard algebra of differentiable functions of  $x \in \mathcal{R}$ . One then concludes that the base space  $\mathcal{R}$  is the standard real line, i.e.,

$$\mathcal{R} = \mathbb{R}. \quad (55)$$

What to be made clear now is the structure of the second sector  $\mathcal{S}_1$ . Note first that  $\varpi_I = \{\xi^r \mid r \in \mathcal{I}\}$  is a finite dimensional complete set. Namely, any function in  $\mathcal{D}_I^q(\mathcal{S})$  can be expanded uniquely by  $\xi^r$ ,  $r \in \mathcal{I}$  and, therefore, the following function, referred to it as “wave function of a point”, can be defined in  $\mathcal{D}_I^q(\mathcal{S})$ :

■ **Definition:** Wave function of a point in  $\mathcal{S}_1$

For a point  $\xi' = e^{i\vartheta'} \in \mathcal{S}_1$ , the folloowing function on the space  $\mathcal{S}_1$  ( $-\pi \leq \vartheta < \pi$ ) can be defined as the wave function of the point  $\xi'$ ,

$$\Psi_{\vartheta'}(\vartheta) := \frac{1}{\sqrt{2\pi N}} \sum_{r \in \mathcal{I}} (\xi^*)^r \cdot (\xi')^r = \frac{1}{\sqrt{2\pi N}} \sum_{r \in \mathcal{I}} e^{ir(\vartheta' - \vartheta)}. \quad (56)$$

Notice that  $\Psi_{\vartheta'}(\vartheta)$  is normalized as

$$\int_{-\pi}^{\pi} |\Psi_{\vartheta'}(\vartheta)|^2 d\vartheta = 1. \quad (57)$$

Then, one can regard

$$\rho_{\vartheta'}(\vartheta) := |\Psi_{\vartheta'}(\vartheta)|^2 \quad (58)$$

as the density of the point  $\xi'$ . With a finite  $N$ , the density (58) indicates that the point  $\xi'$  spreads, i.e., fluctuates around the position  $\vartheta = \vartheta'$  by the quantum effect. In this sense, such a point is called a “fuzzy point. We should remark the feature in the limit  $N \rightarrow \infty$ . When  $N$  goes to infinity, the wave function  $\Psi_{\vartheta'}(\vartheta)$  has a  $\delta$ -functional peak only at  $\vartheta = \vartheta'$ , and so, the point

$\xi'$  locates at the position  $\vartheta = \vartheta'$  with infinite density.

We further remark that the algebra  $\mathcal{D}_I^q(\mathcal{S}_1)$  is just the direct sum of an infinite number of subalgebras which are closed under the action of the difference operator  $\nabla_q$ . As we have done in the preceding subsection, one finds

$$\mathcal{D}_I^q(\mathcal{S}_1) = \bigoplus_{0 \leq \vartheta_0 < \frac{2\pi}{N}} \mathcal{A}_N(\mathcal{S}_{\vartheta_0}^N). \quad (59)$$

Let us observe a subalgebra  $\mathcal{A}_N(\mathcal{S}_{\vartheta_0}^N)$  in detail and, then, make the structure of  $\mathcal{S}_{\vartheta_0}^N$  clear. Taking it into account that the difference operator  $\nabla_q$  generates the finite displacement  $\xi \rightarrow q\xi$  with  $q = e^{i\frac{2\pi}{N}}$ , one sees that  $\mathcal{A}_N(\mathcal{S}_{\vartheta_0}^N)$  is the space of functions on  $N$  fuzzy points as

$$\mathcal{A}_N(\mathcal{S}_{\vartheta_0}^N) = \{\psi(\xi_n) \mid \xi_n = e^{i\frac{2\pi n}{N}} \xi_0, \quad n = 0, 1, \dots, N-1\}. \quad (60)$$

Namely, the internal space  $\mathcal{S}_{\vartheta_0}^N$  is composed of  $N$  fuzzy points,

$$\mathcal{S}_{\vartheta_0}^N = \{\vartheta_n \mid \vartheta_n = \vartheta_0 + \frac{2\pi}{N} n, \ n = 0, 1, \dots, N-1\}, \quad (61)$$

where, instead of  $\xi_n$ , the variables  $\vartheta_n$  defined by  $\xi_n = e^{i\vartheta_n}$  is used for the parametrization of  $\mathcal{S}_{\vartheta_0}^N$ . As we have explained, a point  $\vartheta_n$  extends over the whole  $\mathcal{S}_{\vartheta_0}^N$  according to  $\Psi_{\vartheta_n}(\vartheta)$ , and the distribution of the  $N$  fuzzy points along  $\mathcal{S}_{\vartheta_0}^N$  is given by the superposition

$$\sum_{n=0}^{N-1} \Psi_{\vartheta_n}(\vartheta). \quad (62)$$

Now, putting (45,55) together, we are at the stage to investigate the total space  $\mathbb{R}_q^N$  which is a subspace of  $\mathfrak{R}_N$ , i.e.,

$$\mathbb{R}_q^N := \mathbb{R} \times \mathcal{S}_{\vartheta_0}^N \ni (x, \vartheta_n). \quad (63)$$

The total algebra  $\mathcal{D}(\mathbb{R}) \otimes \mathcal{A}_N(\mathcal{S}_{\vartheta_0}^N)$  is built on  $\mathbb{R}_q^N$ . Before proposing  $\mathbb{R}_q^N$  as a quantum real line, we should make clear the relationship between the external space  $\mathbb{R}$  and the internal space  $\mathcal{S}_{\vartheta_0}^N$ . The difference operator  $\nabla_q$  maps a point in  $\mathbb{R}_q^N$  as

$$\nabla_q : (x, \vartheta_n) \longmapsto (x, \vartheta_{n+1}) \quad (64)$$

where  $(x, \mathcal{P}) \mapsto (x, \mathcal{Q})$  stands for the mapping from the point  $(x, \mathcal{P})$  to the points  $(x, \mathcal{P}) \cdots (x, \mathcal{Q})$  along  $\mathcal{S}_{\vartheta_0}^N$ . By taking the  $N$ -th power of the mapping (64), one finds

$$\nabla_q^N : (x, \vartheta_n) \longmapsto (x, \vartheta_{n+N}). \quad (65)$$

As far as only the internal space  $\mathcal{S}_{\vartheta_0}^N$  is considered,  $\nabla_q^N$  brings a point  $\vartheta_n$  to the same position, since  $\vartheta_{n+N} = \vartheta_n + 2\pi$ . However, once we look over the total space (63), the equivalence  $D_q^N/[N]! \equiv \partial_x$  indicates

$$(x, \vartheta_n + 2\pi) = (x + \epsilon, \vartheta_n) \quad (66)$$

with an infinitesimal number  $\epsilon \in \mathbb{R}$ . Thus, the external space and the internal space are not independent of each other but related as follows: Starting from a point  $x \in \mathbb{R}$  and moving along  $\mathcal{S}^N$ , one reaches the point  $x + \epsilon \in \mathbb{R}$ .

Now, we have understood the structure of  $\mathbb{R}_q^N$ , and it is the time to propose  $\mathbb{R}_q^N$  as a quantum real line when  $q$  is a root of unity.

■ **Proposition 2** : Quantum Real Line  $\mathbb{R}_q^N$



The quantum real line  $\mathbb{R}_q^N$  for the case with the deformation parameter  $q$  at the  $N$ -th root of unity is given by the product of  $\mathbb{R}$  and  $\mathcal{S}_{\vartheta_0}^N$  as

$$\begin{aligned}\mathbb{R}_q^N &= \mathbb{R} \times \mathcal{S}_{\vartheta_0}^N \\ &= \{(x, \vartheta_n) \mid (x, \vartheta_{n+N}) = (x + \epsilon, \vartheta_n)\}\end{aligned}\quad (67)$$

where  $\epsilon$  is an infinitesimal real number and  $n = 0, 1, \dots, N - 1$ .

From the viewpoints of the quantum real line  $\mathbb{R}_q^N$ , each real number  $x$  is furnished with the internal space  $\mathcal{S}_{\vartheta_0}^N(x)$  which is composed of  $N$  fuzzy points  $\vartheta_n$ ,  $n = 0, \dots, N - 1$ . On the other hand, the next point of  $\vartheta_{N-1}$ , i.e.,  $\vartheta_N$  belongs to  $\mathcal{S}_{\vartheta_0}^N(x + \epsilon)$  on  $x + \epsilon$  with an infinitesimal  $\epsilon \in \mathbb{R}$ . Thus, the internal space  $\mathcal{S}_{\vartheta_0}^N(x)$  connects two infinitesimally separated numbers  $x$  and  $x + \epsilon$ . In this sense, the fuzzy internal space  $\mathcal{S}_{\vartheta_0}^N$  will be called, hereafter, an *infinitesimal structure*.

It is interesting to observe the infinitesimal structure from another viewpoint and propose another description of the quantum real line. To this end, we introduce the map  $\tilde{\pi} : \{x\} \times \mathcal{S}_{\vartheta_0}^N(x) \rightarrow \mathcal{M}(x)$  such as

$$\tilde{\pi} : (x, \vartheta_n) \longmapsto \check{x}_n := x \check{+} n\check{\epsilon}, \quad n = 0, 1, \dots, N - 1, \quad (68)$$

where  $\check{\epsilon}$  is some constant and the symbol  $\check{+}$  stands for the sum operation in the space  $\mathcal{M}(x)$ . Then, the space  $\mathcal{M}(x)$  which is defined for each  $x \in \mathbb{R}$  is written explicitly as

$$\mathcal{M}(x) = \{\check{x}_n \mid n = -0, 1, \dots, N - 1, \check{x}_0 \equiv x\}. \quad (69)$$

It should be emphasized that every point  $\check{x}_n$  is fuzzy, i.e., the space  $\mathcal{M}(x)$  is a fuzzy space as well as  $\mathcal{S}_{\vartheta_0}^N$ .

We have now prepared for giving the proposition of another quantum real line when  $q$  is the  $N$ -th root of unity:

■ **Proposition 3** : Quantum Real Line  ${}^\vee\mathbb{R}$

The qantum real line  ${}^\vee\mathbb{R}$  is given by the union of  $\mathcal{M}(x)$  as,

$${}^\vee\mathbb{R} = \bigcup_{x \in \mathbb{R}} \mathcal{M}(x). \quad (70)$$

where  $\mathcal{M}(x)$  is the fuzzy space embedded between two real numbers  $x$  and  $x + \epsilon$  with an infinitesimal real number  $\epsilon \in \mathbb{R}$ .

The quantum real line  ${}^{\vee}\mathbb{R}$  is shown in Fig.2.

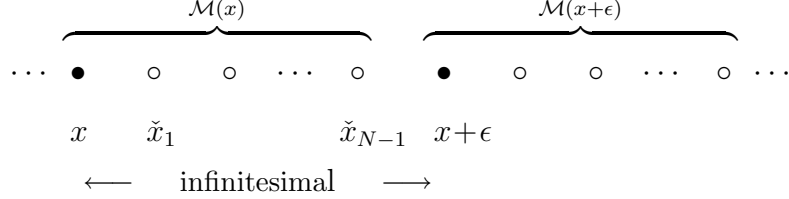


Fig.2 Quantum real line  ${}^{\vee}\mathbb{R}$

where the bullets • stand for the “standard” real numbers  $\mathbb{R}$ , while the circles ○ represent numbers in  ${}^{\vee}\mathbb{R} \setminus \mathbb{R}$ . It is natural to call the space  $\mathcal{M}(x)$  as an “infinitesimal structure”, since  $\mathcal{M}(x)$  lies between  $x$  and  $x + \epsilon$  as shown in Fig. 2. It is quite important to notice the status of the number  $\epsilon$ . As long as we observe  $\epsilon$  within the framework of  $\mathbb{R}$ , it is infinitesimally small. However, from the viewpoint of  ${}^{\vee}\mathbb{R}$ , it is no longer an infinitesimal. Note also that the constant  $\check{\epsilon}$  appeared in (68) is not a real number but an element of  ${}^{\vee}\mathbb{R} \setminus \mathbb{R}$ .

We have obtained two quantum real lines  $\mathbb{R}_q^N$  and  ${}^{\vee}\mathbb{R}$  when  $q$  is the  $N$ -th root of unity. Some remarks are in order. They are, of course, identical with each other under the isomorphism  $\check{\pi}$ . The quantum line  $\mathbb{R}_q^N$  has been introduced as a two-dimensional space. One of the dimensions corresponds to the standard real line  $\mathbb{R}$ , i.e., noncompact and continuous space. The other is  $\mathcal{S}_{\partial_0}^N$  which is compact and fuzzy  $N$ -point space. We have called the extra dimension  $\mathcal{S}_{\partial_0}^N$  the infinitesimal structure. On the other hand, we have defined  ${}^{\vee}\mathbb{R}$  as *an extension* of the standard real numbers  $\mathbb{R}$  by introducing the numbers with “ $\sim$ ”. Due to the extension, the quantum line  ${}^{\vee}\mathbb{R}$  can be viewed as a one-dimensional space. In  ${}^{\vee}\mathbb{R}$ , the infinitesimal structure  $\mathcal{M}(x)$  appears between two adjacent real numbers  $x$  and  $x + \epsilon$ . We should stress that, only for the case when these numbers  $x$  and  $x + \epsilon$  are understood just in the framework of  $\mathbb{R}$ , the interval between them is regarded as of infinitesimally small.

## 5 Concluding Remarks

In this paper, we have investigated possible deformations of the real numbers by taking quantum effects into account and proposed quantum real lines. Upon noticing that the Heisenberg algebra  $\mathcal{L}_H$  spanned by  $a, a^\dagger$  is represented by the algebra of

differentiable functions on  $\mathbb{R}$ , our discussions have started from the deformation of  $\mathcal{L}_H$  according to the Moyal deformation procedure. We have introduced an internal space  $\mathcal{T}_2$  and a functional algebra  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  on  $\mathcal{T}_2$ . The multiplication in  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  is the Moyal product  $*_\gamma$  with respect to the deformation parameter  $\gamma$  and, then,  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  is the noncommutative torus. Although the algebra  $\mathcal{L}_H \times \mathcal{A}(\mathcal{T}_2; *_\gamma)$  can be regarded as a  $q$ -deformed Heisenberg algebra with the deformation parameter  $q = e^{I\gamma}$ , we have further derived the algebra  $\hat{\mathcal{L}}_q := \mathcal{L}_H \times \hat{\mathcal{A}}(\mathcal{T}_1)$  from  $\mathcal{L}_H \times \mathcal{A}(\mathcal{T}_2; *_\gamma)$  by reducing the base space  $\mathcal{T}_2$  to the one-dimensional space  $\mathcal{T}_1$ . The crucial point here is that, due to the reduction  $\mathcal{T}_2 \rightarrow \mathcal{T}_1$ , the functional algebra  $\mathcal{A}(\mathcal{T}_2; *_\gamma)$  becomes the operator algebra  $\hat{\mathcal{A}}(\mathcal{T}_1)$ . We have treated  $\hat{\mathcal{L}}_q$  as our  $q$ -deformed Heisenberg algebra.

The algebra  $\hat{\mathcal{L}}_q$  has been represented by the algebras of  $q$ -differentiable functions on the base spaces  $\mathfrak{R}$  and  $\mathfrak{R}_N$ , i.e., the algebra  $\mathcal{D}^q(\mathfrak{R})$  for the case where  $q$  is not a root of unity and  $\mathcal{D}_N^q(\mathfrak{R}_N)$  for the case where  $q$  is the  $N$ -th root of unity. Focussing on the  $q$ -differential structures of these algebras, the geometrical structures of the base spaces  $\mathfrak{R}$  and  $\mathfrak{R}_N$  have been deduced. Finally, we have proposed the quantum real line  $\mathbb{R}_D$  from  $\mathfrak{R}$  and lines  $\mathbb{R}_q^N$  and  ${}^\vee\mathbb{R}$  from  $\mathfrak{R}_N$ .

Let us summarize the features of these quantum lines.

- The quantum real line  $\mathbb{R}_D$

It is a discrete space composed of  $\hat{\chi}_n$ ,  $n \in \mathbb{Z}$ . A quantum real number in  $\mathbb{R}_D$  is localized upon a spot and distance between two adjacent spots is specified by  $\gamma$ .

- The quantum real line  $\mathbb{R}_q^N$

It is a two-dimensional object as  $\mathbb{R}_q^N = \mathbb{R} \times \mathcal{S}_{\vartheta_0}^N \ni (x, \vartheta_n)$  where  $\mathcal{S}_{\vartheta_0}^N$  is composed of  $N$  fuzzy points. The fluctuation of a point  $\vartheta_n$  along the internal space  $\mathcal{S}_{\vartheta_0}^N$  is described by the wave function  $\Psi_{\vartheta_0}(\vartheta)$  given in (56). The important fact is that moving around  $\mathcal{S}_{\vartheta_0}^N$  generates an infinitesimal displacement along  $\mathbb{R}$ .

- The quantum real line  ${}^\vee\mathbb{R}$

It is isomorphic to  $\mathbb{R}_q^N$ . However, in defining  ${}^\vee\mathbb{R}$ , we have extended the concept of the standard real number, i.e.,  ${}^\vee\mathbb{R}$  is composed of the extended real numbers  $\check{x}_n = \check{x}_0 + n\check{\epsilon}$ ,  $n = 0, 1, \dots, N-1$  where  $\check{x}_0 \equiv x \in \mathbb{R}$  and  $\check{\epsilon} \notin \mathbb{R}$  is the unit of  ${}^\vee\mathbb{R}$ . Thus, each real number  $x \in \mathbb{R}$  possesses the infinitesimal structure  $\mathcal{M}(x) = \{\check{x}_0, \check{x}_1, \dots, \check{x}_{N-1}\}$  between  $x$  and  $x + \epsilon$ . An important fact is that  $\epsilon$  is

treated as an infinitesimal within  $\mathbb{R}$ , while it is not within  ${}^{\vee}\mathbb{R}$ . The  $N$ -point space  $\mathcal{M}(x)$  is a fuzzy space as well as  $\mathcal{S}_{\vartheta_0}^N$ . the infinitesimal structure of  $x$ .

It is interesting to consider the *classical* limit  $q \rightarrow 1$  in each quantum line. For the line  $\mathbb{R}_D$ , the limit corresponds to the situation where the parameter  $\gamma$  is so small that the distance  $\hat{\chi}_{n+1} - \hat{\chi}_n \sim \gamma$  shrinks to zero. Furthermore, in the limit, the difference operator  $\Delta_\gamma$  returns to the standard differential operator. Therefore,  $\mathbb{R}_D$  goes back to a continuous and smooth one-dimensional line, i.e., the standard real line  $\mathbb{R}$ . In other words, the limit corresponds to the situation where the wave length  $\lambda$  to observe  $\mathbb{R}_D$  is so long  $\lambda \gg \gamma$  that the discreteness of the quantum line is not visible.

The classical limits of the lines  $\mathbb{R}_q^N$  and  ${}^{\vee}\mathbb{R}$  are to be noticed. In these lines,  $q \rightarrow 1$  is realized by taking  $N = 1$  or  $N \rightarrow \infty$ . When  $N = 1$ ,  $\mathcal{M}(x)$  becomes the one-point space  $\mathcal{M}(x)|_{N=1} = \{\tilde{x}_0 = x \in \mathbb{R}\}$ , i.e., the infinitesimal structure vanishes and  ${}^{\vee}\mathbb{R} = \mathbb{R}$ . On the other hand, when  $N \rightarrow \infty$ , the space  $\mathcal{M}(x)$  becomes a continuous space. Similarly, for the line  $\mathbb{R}_q^N$ , the internal space  $\mathcal{S}_{\vartheta_0}^N$  becomes a one-point space when  $N = 1$  and the smooth circle  $S_1$  when  $N \rightarrow \infty$ . However, for the limit  $N \rightarrow \infty$ , there remains a room to discuss whether  $\mathcal{M}(x)$  and  $\mathcal{S}_{\vartheta_0}^N$  expand infinitely large or keep small. If  $\mathcal{M}(x)$  extends to infinity, then  $\mathcal{M}(x)$  itself is raised to  $\mathbb{R}$ . On the contrary, if the size of  $\mathcal{S}_{\vartheta_0}^N$  remains small,  ${}^{\vee}\mathbb{R} \rightarrow \mathbb{R} \times S_1$ , i.e., two-dimensional cylinder appears!

Let us end this paper by giving a comment on another limit for  ${}^{\vee}\mathbb{R}$ . It should be noticed that it is possible to take the limit  $\tilde{\epsilon} \rightarrow 0$  with keeping  $N$  finite. In this case,  $\epsilon$  also goes to 0. Then, it seems that  ${}^{\vee}\mathbb{R}$  reduces to  $\mathbb{R}$ . However, there is a crucial difference, i.e., upon defining

$$\widehat{\mathbb{R}}_N := \lim_{\tilde{\epsilon} \rightarrow 0} {}^{\vee}\mathbb{R}, \quad (71)$$

$\widehat{\mathbb{R}}_N$  is also an extension of  $\mathbb{R}$ . Each point in  $\widehat{\mathbb{R}}_N$  is degenerated by  $N$  number of points  $\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{N-1}$ . In other words,  $\widehat{\mathbb{R}}_N$  is regarded as a bundle of  $N$  real lines. What happens to a function  $\Phi$  on  ${}^{\vee}\mathbb{R}$  in this limit? Since  ${}^{\vee}\mathbb{R}$  looks like a bundle of  $N$  real lines and  $\tilde{x}_r$  becomes a point on the  $r$ -th line, one finds the reduction  $\Phi(\tilde{x}_r) \rightarrow \phi_r(x)$ . Here,  $\phi_r(x)$  is regarded as a function living on the  $r$ -th real line. Thus, in the limit  $\epsilon \rightarrow 0$ , function  $\Phi$  can be viewed as a unification of  $N$  functions  $\phi_i$  on  $\mathbb{R}$  as  $\Phi \rightarrow \phi_0 \oplus \phi_1 \oplus \dots \oplus \phi_{N-1}$ .

An important application will be found when  $N = 2$ . In this case, the space  $\mathcal{M}(x)$  is the two-point space such as  $\mathcal{M}(x) = \{\bullet, \circ\}$ . In the limit  $\epsilon \rightarrow 0$ ,  $\widehat{\mathbb{R}}_2$ , i.e., a pair of two real lines. It is interesting to expect that supersymmetric models are described in our formalism. In explicit, one of the real lines in  $\widehat{\mathbb{R}}_2$  corresponds to the space where bosons  $\phi_B$  live and on the other line, fermions  $\psi_F$  live. Namely, field  $\Phi$  can be regarded as a superfield  $\Phi \rightarrow \phi_B(x) \oplus \psi_F(x)$ . The above prospect follows from the facts: Actions of the generator of supertransformation  $\delta_S$  move, e.g.,  $\phi_B(x)$  as  $\phi_B(x) \xrightarrow{\delta_S} \psi_E(x) \xrightarrow{\delta_S} \phi_B(x + \epsilon)$ . Similarly, in our model with  $N = 2$ , we have seen the movements  $(x, \circ) \xrightarrow{D_q} (x, \bullet) \xrightarrow{D_q} (x + \epsilon, \circ)$ . Furthermore, the author has shown the equivalence between the quantum universal enveloping algebra  $U_{qsl}(2, \mathbb{C})$  with  $q$  at the 2-nd root of unity and the supersymmetric algebra  $\text{Osp}(2|1)$  [4]. Thus, one expects that quantization of geometry is the origin of the supersymmetry. The investigation will appear elsewhere.

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